Competing Meanings, Perturbation, and Engendering Shifts in (Prospective) Teacher Meanings

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Moore, K. C., Tasova, H., Stevens, I. E., & Liang, B. (2024). Competing meanings, perturbation, and engendering shifts in (prospective) teacher meanings. In K. W. Kosko, J. Caniglia, S. A. Courtney, M. Zolfaghari, & G. A. Morris (Eds.), *Proceedings of the 46th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1145-1155). Kent State University.

Available at:
https://www.pmena.org/pmenaproceedings/PMENA%2046%202024%20Proceedings.pdf

COMPETING MEANINGS, PERTURBATION, AND ENGENDERING SHIFTS IN (PROSPECTIVE) TEACHER MEANINGS

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Quantitative reasoning's emergence as a foundation for students' mathematical development has generated a need for supporting teachers' capacity to teach for such reasoning. In this paper, we discuss a meanings perspective on working with prospective and practicing teachers in order to support their constructing meanings that foreground quantitative reasoning. Our meanings perspective, referred to as competing meanings, involves a problematization of extant meanings, the construction of alternative meanings, and a critical comparison of each. Here, we present our perspective and informing theories. We also draw on our empirical work to provide tangible and research-based examples of our competing meanings perspective.

Keywords: Cognition, Preservice Teacher Education, Teacher Knowledge, Learning Theory.

Students' quantitative reasoning refers to the ways in which students conceive of and reason with measurable attributes constituting some phenomenon or context (Smith III & Thompson, 2007; Thompson, 2011; Thompson & Carlson, 2017). Addressing number concepts, fractional reasoning, proportional reasoning, algebraic reasoning, rate of change concepts, and function concepts (e.g., Karagöz Akar et al., 2022; Steffe & Olive, 2010; Thompson & Carlson, 2017), researchers have identified quantitative reasoning as a bedrock for students' mathematical development. These same researchers have highlighted that the various factors influencing students' educational experiences do not sufficiently engender or support students' quantitative reasoning. Whether with respect to improved curricular materials, continued knowledge development, or targeted pedagogical practices, a pressing need is better understanding how to prepare teachers in supporting their students' quantitative reasoning.

Over the past decade-plus we have engaged in a research program to understand not only students' quantitative reasoning, but also that of prospective and practicing teachers. Our primary research emphasis has been understanding the relationship between teachers' mathematical meanings and their quantitative reasoning, including how to engender teachers' quantitative reasoning so that it might be leveraged to support shifts in their meanings. We have specifically sought to support shifts reflecting those meanings identified by researchers as critical for K-16 students' mathematics. We report on a perspective for supporting such shifts in this paper.

We term our perspective *competing meanings* due to its simultaneous focus on teachers' extant meanings, the meanings we seek to engender and center when working with teachers, and interactions between those meanings we seek to provoke. In what follows, we first provide background theory that informs our perspective including Piaget's epistemology (Piaget, 2001), Thompson’s theory of meaning and quantitative reasoning (Thompson, 2016; Thompson & Carlson, 2017), and Harel’s notion of intellectual need (Harel, 2013). Drawing on those informing theories, we outline the perspective of competing meanings as it relates to working with teachers, whether prospective or practicing. We support an operational approach to competing meanings by also providing a tangible example of it, both in the context of task design and research participant themes. We close with potential implications and future work by drawing specific attention to areas of theory left to flesh out or connect with.

Informing Theory and Background

Our perspective is informed by Piaget’s genetic epistemology, including von Glasersfeld’s (1995) extension of it. We focus here on the constructs of *assimilation*, *perturbation, accommodation*, and *equilibration*, and we point the reader to Dawkins et al. (2024) for an extensive collection of Piaget’s theory in mathematics education. Assimilation is the process by which an individual conceives a present experience via their current conceptual structures (von Glasersfeld, 1995). It is a constructive process that shapes an experience so that it affords and is constituted by those structures. In some cases, assimilation to extant conceptual structures results in an unexpected experience, which engenders a state of perturbation (von Glasersfeld, 1995). A perturbation can stem from several causes. For one, an individual might obtain an unexpected result after enacting a conceptual structure, thus yielding a sense of perplexity. As another example, in enacting a conceptual structure, an individual might become aware of some experiential feature that leads to their questioning the efficacy or relevance of that structure.

Having experienced a perturbation, an individual engages in activity to reconcile that cognitive state. One form of reconciling a perturbation involves affective and coping responses, such as anxiety leading to disengagement (Tallman & Uscanga, 2020). Another form of reconciliation is that of accommodation, which can take on several forms. To name a few, the conceptual structure used in assimilation could be modified, an alternative conceptual structure could be enacted, or a novel conceptual structure could be constructed (von Glasersfeld, 1995). Regardless, accommodation is an act of learning via the elimination of a perturbation through a cognitive construction or reorganization. It often entails sustained, and effortful, cognitive engagement. Piaget hence referred to the process of accommodation as one of equilibration that establishes a cognitive state of equilibrium (von Glasersfeld, 1995).

In service of operationalizing the aforementioned Piagetian constructs, Thompson introduced the intertwined theories of quantitative reasoning (Thompson, 2011) and meaning (Thompson, 2016). With respect to the latter, Thompson’s (2016) theory of meaning is rooted in Piaget’s genetic epistemology and refers to an organization of operations, images, and other meanings. As it relates to the act of teaching, Thompson’s theory of meaning is connected to that of Silverman and Thompson (2008), who outlined a developmental process that spans the construction of personalized knowledge to the transformation of that knowledge to incorporate student meanings and pedagogical implications. That is, Silverman and Thompson recognized the importance of teachers’ mathematical meanings including teachers’ construction of *key developmental understandings*, which are understandings critical to the development of coherent and generative mathematical concepts (Simon, 2006). Before using Thompson’s theory of quantitative reasoning to further illustrate this perspective, we note that the perspective emphasizes mathematical knowledge as a dynamic, in-the-moment implicative base of *knowing* for action, as opposed to a static, declarative base of *knowledge* for action (Liang, 2021; Thompson, 2016).

Thompson's (2011, 2012) theory of quantitative reasoning provides one framework for situating Piaget's genetic epistemology, meaning, and key developmental understandings. *Quantitative reasoning* is reasoning that involves conceiving situations in terms of measurable attributes (i.e., quantities) and relationships between those attributes (i.e., quantitative relationships). Quantitative relationships form the basis for the construction and abstraction of mathematical objects (Moore et al., 2022; Smith III & Thompson, 2007). *Covariational reasoning* is a particular form of quantitative reasoning that involves constructing and coordinating quantities that vary in tandem (Carlson et al., 2002; Saldanha & Thompson, 1998; Thompson & Carlson, 2017). A growing number of researchers have identified important nuances in student and teacher thinking in this area (see Karagöz Akar et al., 2022 for a collection of contributions and researchers). Using the framework by Carlson et al. (2002), one meaning entailing covariational reasoning involves assimilating a situation via *directional* and *amounts of change* relationships. For instance, Ellis et al. (2015) explored students' meanings for exponential relationships in the situation of (magic) plant growth and the quantities height and time. The students' meanings involved their constructing the directional covariation of quantities (e.g., as time increases, height increases), and coordinating additive changes in one quantity with multiplicative changes in the other (e.g., as time increases additively, height increases by increasing amounts while preserving a constant ratio for a constant time period). Here, the operations constituting the meaning for exponential relationships involve conceiving the variation in each quantity, coordinating those two variations to construct and compare changes in each, and considering how the constructed covariational relationship is relevant to different contexts (e.g., a growing plant, a Cartesian graph, or a table).

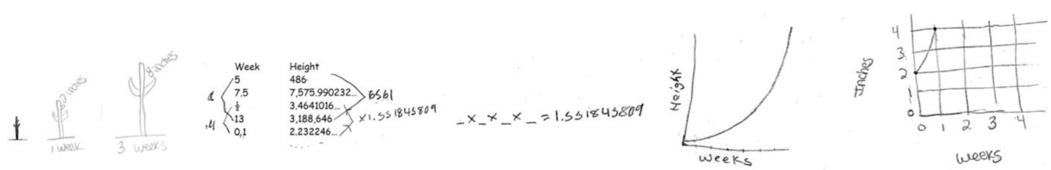

Figure 1: Students' coordinating height and time (Ellis et al., 2015, pp. 143, 147, and 149)

Our work is also informed by Harel's (2013) *intellectual need*. We use intellectual need to clarify the perturbations targeted by our competing meanings perspective. Harel defined intellectual need as "a perturbational state resulting from an individual's encounter with a situation that is incompatible with, or presents a problem that is unsolvable by, his or her current knowledge" (2013, p. 122). Importantly, Harel's intellectual need refers to a state of perturbation that affords learning, and is thus not merely a state of confusion. A researcher is positioned to claim an individual has experienced an intellectual need when the meanings needed to reconcile an experienced perturbation are within the individual's zone of proximal development, whether that development be in the context of reasoning or domain practices (Harel, 2013; Weinberg et al., 2023). With respect to the work here, intellectual need orients us toward not only seeking to engender perturbations, but also having in mind the ways in which teachers' available reasoning can act as an asset in reconciling that perturbation. Furthermore, intellectual need draws our attention to forms of perturbations that promote their reflective comparison of meanings that caused the perturbations and those that reconciled those perturbations.

In summary, Piaget's and von Glasersfeld's framings of knowing provide us guiding cognitive mechanisms. Thompson's perspective on meaning, with Silverman, Thompson, and Simon's descriptions of how meanings inform teaching, further clarify our attention to the ways an individual's personal meanings may be organized and transformed so that they are generative and flexible during the act of teaching. Theories of quantitative and covariational reasoning provide us concrete constructs by which to specify and differentiate mathematical meanings. Lastly, Harel's notion of intellectual need aids us in clarifying the type of perturbations we seek to engender with teachers. Namely, we focus on perturbations that necessitate the enactment of alternative meanings to reconcile them (i.e., equilibration via accommodation). Furthermore, we focus on the transformative learning experiences that occur when a process of equilibration is accompanied by a subsequent perturbation that motivates reflectively comparing meanings.

Competing Meanings

Our competing meanings perspective identifies one form of learning via particular forms of intellectual need and, hence, perturbation and accommodation. Stated generally, the competing meanings perspective includes an individual experiencing a problematized extant meaning; enacting an alternative meaning; and, through additional processes of perturbation and accommodation, comparing the extant meaning and alternative meaning (Figure 2).

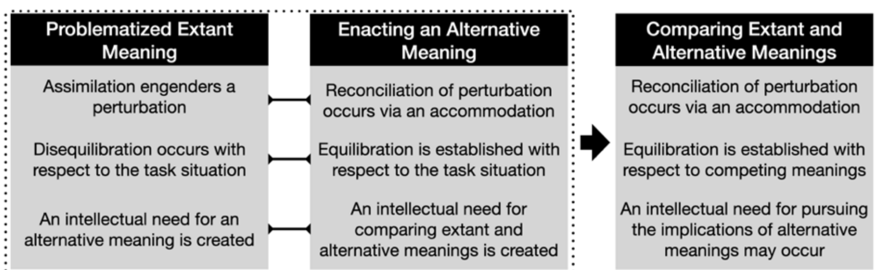

Figure 2: The competing meanings perspective

A problematized extant meaning first occurs via an act of assimilation that engenders a perturbation and an intellectual need for an alternative meaning. Then, via enacting that alternative meaning, the individual reconciles their perturbation with respect to the task situation associated with the initial perturbation. Critical to the competing meanings perspective is that a subsequent state of perturbation then occurs. Whereas the initial intellectual need was respect to the goal-oriented activity of the task, a subsequent round of intellectual need is created at the level of meanings; the individual becomes perplexed as to why their extant meaning results in a perturbation while the alternative meaning does not. The disparate nature of the meanings is thus at the root of the perturbation and associated intellectual need. By disparate, we mean that, *in that moment*, the individual infers that their two held meanings entail important differences and incompatibilities that are not trivial to resolve. This perplexity positions the individual to take each meaning as an object of thought and hold them against each other (i.e., competing meanings) in order to reconcile that perturbation. Yet an additional intellectual need might result from this process, motivating the individual to explore the implications of that reconciliation, particularly if the alternative meaning is novel and viewed as a potentially more productive meaning. We return to this point in the closing of the paper.

To situate and illustrate the perspective, we start with an abstract and abridged example. Consider a hypothetical student or teacher, who we name Blinder. For a particular concept, Blinder has constructed a meaning that we denote by M_a (we remind the reader that a meaning might be composed by a system of meanings), which has served as productive throughout his schooling experience. Entering our class or professional development, we might intend, for a variety of reasons, that Blinder construct an alternative meaning. We denote this alternative meaning by M_b. In working with Blinder, we determine that he holds meaning M_a and that meaning M_a and M_b are disparate; M_a is not a foundational way of thinking for M_b and, in fact, can inhibit the construction of and ability to teach for M_b. This raises the question: how do we engage with Blinder in a way that honors M_a and affords constructing M_b? This is a situation we have been presented with frequently in research, teaching, and professional development settings with students and teachers (e.g., Moore, Stevens, et al., 2019; Tasova, 2021).

Using Linearity to Illustrate the Competing Meanings Perspective

Consider linear relationships as an example topic to contextualize the abstract presentation above. Our work has adopted a quantitative reasoning perspective to center a meaning for linear relationships that involves constructing a constant rate of change. A constant rate of change between two quantities means that as the quantities' magnitudes covary, their amounts of change exist in a proportional relationship. For any arbitrary change x (e.g., Δx), y changes by a scaler factor m of that change (e.g., $m \cdot \Delta x$). If that arbitrary Δx is then scaled by a factor c, the change in y is scaled by the same factor, yielding a corresponding change in y of $c \cdot m \cdot \Delta x$. This is a critical and productive meaning (i.e., M_b), yet our and others' work with teachers and students suggest that this is not always a typical meaning (Byerley & Thompson, 2017; Lobato et al., 2003; Moore, Silverman, et al., 2019; Thompson & Thompson, 1996; Zaslavsky et al., 2002).

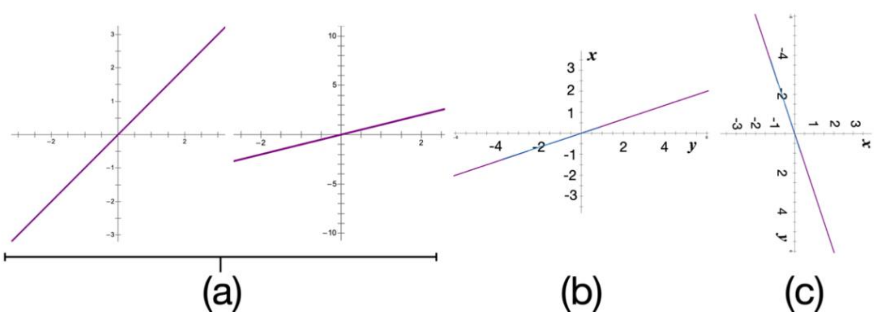

Figure 3: (a) Two graphs of $y = x$ and (b-c) two graphs of $y = 3x$.

A common extant meaning (i.e., M_a) for linear relationships is shaped-based (Ellis & Grinstead, 2008; Moore, 2021; Moore, Stevens, et al., 2019; Nagle & Moore-Russo, 2013; Zaslavsky et al., 2002), which entails reasoning about linear relationships in terms of properties of slope like movement and direction in association with learned formulas (e.g., $(y_2 - y_1)/(x_2 - x_1)$). These associations are forms of declarative knowledge, as opposed to symbolizing abstracted covariational relationships. An example of this is an individual comparing the visual steepness of two lines to conclude the former has a greater rate of change than the latter (Figure 3a). As another example of this meaning, an individual could conceive Figure 3b as having an incorrect rise-and-run and Figure 3c as a negative slope or rate of change because of its downward, left-to-right direction (Moore, Silverman, et al., 2019; Moore, Stevens, et al., 2019).

Returning to the question raised above, in working with individuals holding a shape-based meaning, M_a, as their dominant meaning, we intend to both honor those shape-based meanings while determining how to support their constructing a rate of change meaning, M_b. Our goal is also to support their constructing M_b so that it becomes a meaning they view as important and productive (for them and their students), and more so than that of the shape-based meaning. Before describing an approach that draws on the competing meanings perspective, we recognize one way to support M_b is to use tasks in which M_a is not relevant, but M_b is. Similarly, one might use tasks that target M_b through focused, closed-ended questions and design. In our experience, such tasks are useful to engender M_b and possibly draw connections with M_a. Yet, such tasks can be so contrived as to feel too disjoint from the classroom for teachers. Relatedly, those tasks do not problematize M_a and generate an intellectual need for M_b so that the latter becomes their predominant or habitual meaning. With respect to teachers, for the tasks they envision teaching, M_a remains just as relevant, is more familiar or habitual, and is often more cognitively efficient. Our solution to this issue is to use tasks that not only afford or target M_b, but also draw the meanings M_a and M_b into competition with each other. Doing so requires that the task is designed so that M_a is still relevant to the task. Furthermore, we intend the enactment of M_a to lead to a conclusion that not only invites further thought, but that also stands in opposition to the conclusion derived from enacting M_b. This underscores the competing aspect of competing meanings. The initial perturbation should not leave the teacher viewing M_a as entirely problematic or unrelated, as it is through viewing M_a as still relevant despite some perturbation that the individual is positioned to compare its viability against that of M_b.

We use the graph in Figure 3b to illustrate how we have attempted to target the cognitive process in Figure 2 and draw meanings into competition with each other in the context of linear relationships. When working with teachers, we present this task in two parts. We first provide the graph as illustrated in Figure 3b, but *without* the axes-labels "x" and "y". We explain that a student provided the graph (without labels) as a solution to graphing "$y = 3x$", and we ask them to consider how the student might have been thinking. After the teacher has exhausted the number of ways they can hypothesize as to how the student might have been thinking (see Moore, Silverman, et al., 2019 for examples), we then provide Figure 3b with "x" and "y". We explain that the student added the labels to clarify their solution. We ask the teacher to comment on the graph, and we conclude the task asking how they would respond to the student as their teacher. We note that Figure 3c is created by most teachers when making sense of the solution due to their rotating the graph to horizontally orient x. If the teacher does not rotate the graph, we rotate the graph and ask them to consider it in that orientation, as well.

The task incorporates the competing meanings perspective by using the following principles: (a) it sensibly affords assimilation to M_a and M_b; (b) in the event that M_a is enacted, it is likely to result in a perturbation, but still be viewed as relevant to the task; (c) in the event that M_a engenders a perturbation, M_b is likely available to the student or within their zone of proximal development; (d) the enactment of M_b can reconcile a perturbation stemming from M_a; (e) the teacher has the opportunity to reflectively compare the affordances and constraints of M_a and M_b;

and, critically, (f) the teacher is likely to perceive the task, M_a, and M_b as relevant to their instruction. Relating these task features to the cognitive account in Figure 2, (a) and (b) occasion problematizing an extant meaning; (a), (c), and (d) relate to accommodation via the enactment of an alternative meaning; and (b), (d), (e), and (f) support reflecting on and comparing extant and alternative meanings. Furthermore, the task embodies the *competing* aspect of competing meanings by using a situation in which M_a and M_b yield sensible, yet different conclusions. Enacted as is, M_a results in classifying the solution and its rotated version (Figure 3c) as inaccurate representations of $y = 3x$ (e.g., the "slope" is wrong in Figure 3b and 3c), while M_b affords accepting both as accurate (e.g., each is the set of points so that y is three times as large as x and for any change in x, y changes by three times that amount). This in-the-moment incompatibility aids comparing the generativity and generalizability of each meaning including weighing which is better viewed as derivative of the other (e.g., slope as an implication of rate of change is more generative and generalizable than rate of change as an implication of slope).

Data Illustrations

Although this is chiefly a theoretical report focused on a particular form of learning and cognitive activity, it represents generalizations from a collection of empirical studies with students and teachers. The studies and their methodologies entailed semi-structured clinical interviews (Ginsburg, 1997) and various forms of teaching experiments (Steffe & Thompson, 2000), and are summarized in Moore et al. (2022) and Moore et al. (2024). Here, we draw from our empirical data with prospective teachers working the aforementioned task.

We use Table 1 to provide emblematic examples of each competing meanings component presented in Figure 2. Due to space constraints, we use quotes and only a brief narrative situating those quotes. We point the reader to our work referenced above for more detailed narratives of the students' actions and meanings. With respect to a problematized extant meaning, the example quote is from a participant, Lizzie, who conceived Figure 3b as having a "positive slope" and Figure 3c as having a "negative slope" due to their direction of rise and run (i.e., M_a). For both graphs, Lizzie checked points to verify the accuracy of the formula $y = 3x$. This, when paired with the slope discrepancy between the given and rotated graph, left her perturbed and calling into question the viability of her thinking on the task ("this is so annoying"). With respect to enacting alternative meanings (i.e., M_b), Tatiana's quote illustrates that by conceiving the graph via quantitative and covariational operations, she determined the graph to be a viable representation of $y = 3x$. This occurred after having not determined what was to her a satisfactory way to produce the unlabeled graph. In attributing a viable way of reasoning to the student solution, it also supported her reflecting on that meaning in terms of its flexibility. This is a key foundation for the reflective comparison of meanings.

The problematization of an extant meaning can occur in a reflexive process with the enactment of alternative meanings. Similarly, the phenomenon of reflectively comparing extant and alternative meanings does not always immediately follow that process. It more often occurs iteratively across a sequence of tasks. With that said, the provided quote is from Ada and it occurred after engaging in several tasks across an instructional sequence. It illustrates that by comparing extant and alternative meanings, she came to view rate of change as a dominant meaning. She conceived slope as a visual property (i.e., M_a) derivative of and thus subordinate to rate of change (i.e., M_b). This enabled her to consider a graph like that in Figure 3c as having a rate of change of 3 and, hence, a positive slope. Furthermore, she could couch her appraisal of the graph in terms of differentiating between the underlying mathematical concept of rate of change and common communicative practices (i.e., conventions). Such appraisals are critical to learning mathematics (Moore, Silverman, et al., 2019). As the participant Thomas described when privileging rate of change, "it's smart [of a student] to understand that it's not glued."

Table 1: Quotes Associated with each Competing Meanings Component

Component	Graph Considered	Quote
Problematized Extant Meanings	Figures 3b-c (with labels)	Lizzie: I'm rising this three…then I'm running negative one…the slope is negative again…this is so annoying.
Enacting Alternative Meanings	Figure 3b (with labels)	Tatiana: Oh…we have a clever kid over here…so it now technically is y equals three x…not the standard way of doing it…They see the relationship between x and y.
Compare Extant and Alternative Meanings	Figure 3c (with labels)	Ada: …even though it looks like a negative slope…we call it slope because it's visual and it's easy to visualize a negative and positive slope. But that's only visual on our conventions of how we set it up…slope is rate of change, we can still see that for like equal increases of x we have an equal increase of y of three. And so for equal positive increase of one we have an equal positive increase of three. And so, it is a positive slope.

Closing

We presented one learning form that identifies how two meanings might be brought into comparison via processes of assimilation, accommodation, and perturbation. We illustrated how such a process involves different forms of intellectual need, including that with respect to solving a task, comparing meanings, and considering the implications of those meanings. We also illustrated the competing meanings perspective through a task and emblematic participant activity. The competing meanings perspective is still in its infancy as a construct. Moving forward, we envision a need for further connecting to other extant constructs and perspectives, the results of which will continue to shape and develop the idea of competing meanings.

We have concentrated much of our research focus on the first two aspects competing meanings and relatively less on the nuanced ways in which teachers compare extant and alternative meanings (cf. Paoletti, 2020). A reflective comparison of meanings is a developmental process that occurs across a sequence of experiences, and it is through such a process that key developmental understandings are constructed and associated pedagogical implications are anticipated (Silverman & Thompson, 2008; Simon, 2006). We envision a fruitful area of inquiry to be more detailed investigations into how the competing meanings perspective might be used to engender such reflective comparisons and, accordingly, the construction of key developmental understandings. Furthermore, we view a need for further relating this process to the development of mathematical knowledge for teaching. Here, we do not make strong claims regarding the development of mathematical knowledge for teaching. There are numerous factors other than a teacher's knowledge that mitigates their instructional practices and the meanings they target in their classroom. But, literature identifies the critical role of meanings and teachers being reflectively aware of them (Liang, 2019, 2023; Tallman & Frank, 2020; Thompson, 2016), and the competing meanings perspective is one potential tool to support the transformation of knowledge to that which informs instructional action.

For the purpose of adhering to the space constraints of the current report, we situated our work in the theories that directly informed its emergence and development. There is significant literature on learning, conceptual change, and perturbation, and thus an additional need is to further situate the notion of competing meanings within that literature. For example, Vinner and Dreyfus (1989) proposed *compartmentalization* as the phenomenon in which a learner has two potentially conflicting meanings. Noah-Sella et al. (2022) have since extended this phenomenon to incorporate Thompson's theory of meaning and explore calculus students' integral meanings. Their perspective foregrounds cases in which a researcher perceives a potential conflict or relationship between meanings, but the participant does not. The competing meanings perspective might contribute a way by which one considers how to support a student or teacher in bringing that conflict to the surface. As another example, researchers have productively pursued characterizing learning using Piaget's forms of reflective abstraction (Ellis et al., 2024; Simon et al., 2010; Tallman & O'Bryan, 2024), including theorizing its role in constructing mathematical knowledge for teaching (Liang, 2021, 2023). We envision that drawing connections with this work will provide insights into how aspects of the competing meanings perspective are related to crucial abstraction processes.

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