Examining Complex Numbers with Learning Through Activity

Gulseren Karagöz Akar
Kevin C. Moore

Karagöz Akar, G., & Moore, K. C. (2024). Examining complex numbers with Learning Through Activity. In K. W. Kosko, J. Caniglia, S. A. Courtney, M. Zolfaghari, & G. A. Morris (Eds.), *Proceedings of the 46th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1599-1604). Kent State University.

Available at:
https://www.pmena.org/pmenaproceedings/PMENA%2046%202024%20Proceedings.pdf

EXAMINING COMPLEX NUMBERS WITH LEARNING THROUGH ACTIVITY

Gülseren Karagöz Akar
Boğaziçi University
gulseren.akar@boun.edu.tr

Kevin C. Moore
University of Georgia
kvcmoore@uga.edu

Simon and his colleagues' (2010, 2018) development of Learning Through Activity (LTA) offered a theory for explaining a mechanism for mathematics conceptual learning and an approach to the instructional design to foster it. LTA was an empirically based framework that was developed studying mainly rational number concepts. Taking a step forward, in this paper, we elaborate on and exemplify LTA instructional design principles using an advanced mathematics topic. For this purpose, we share an articulation of the Cartesian form of complex numbers as a mathematical concept and a task sequence to learn this concept. Also, we share an example of the reflective abstraction of this concept with data from teaching experiments with a prospective secondary mathematics teacher. Providing ways to utilize the LTA framework for an advanced mathematical topic, we discuss implications for teaching and learning.

Keywords: Mathematics Learning, Learning Through Activity, Complex Numbers

Conceptual mathematics learning for all students is the main goal of mathematics instruction (Common Core School Mathematics, 2010). Planning effective instructional designs is at the heart of conceptual learning (Gravemejier, 2004; NCTM, 2000; Simon & Tzur, 2004). Over the last three decades, Simon postulated the construct of hypothetical learning trajectory (HLT) for describing key aspects of planning mathematics lessons for promoting conceptual learning from a constructivist perspective (Simon, 1995) and extended it by postulating a mechanism for mathematics learning and instructional design principles for students' learning of mathematics through their own activity (Simon & Tzur, 2004; Simon et al., 2010; Simon et al., 2018).

Learning Through Activity (LTA) design principles deepen and extend HLT steps by further explicating the reciprocal relationship between tasks and learning goals. LTA design principles allow the teacher, first, to delineate "..an activity that students have currently available that can be the basis for the abstraction specified in the learning goal (Simon et al., 2018, p. 104). The activity refers to students' two or more goal-directed mental actions (Simon et al., 2018). Secondly, the teachers design or choose a task sequence intended to foster the particular activity and abstraction on the part of students. The tasks based on LTA design theory are as such: "… the learner can already solve that causes the learner to coordinate actions corresponding to prior concepts in such a manner that the intended new concept is inherent in this coordination" (Dreyfus, 2018, p. 217). In addition, the tasks neither explain nor mention the intended learning goal; however, they have the underlying goal of leading the students to grasp the logical necessity of the learning goal on their own (Dreyfus, 2018). Therefore, for both research and teaching purposes, researchers took attention to two important aspects of the instructional design: 1) an awareness and articulation of the learning goals through students' own activity, and 2) the creation or choice of tasks that have the potential to foster such activity based on which mathematics conceptual learning might be promoted (e.g., Dreyfus, 2018; Simon et al., 2018; Tzur, 2018). Further research is suggested to investigate the applicability of LTA to other levels of mathematical learning (e.g., advanced mathematics) (Dreyfus, 2018; Simon et al., 2018). Taking this as a challenge, in this paper, we elaborate on and exemplify LTA instructional design principles to promote conceptual learning of an advanced mathematics concept, the Cartesian form of complex numbers. Thus, we attempted to answer the following theoretical and empirical questions: What might be a possible example of LTA design principles used with respect to learning goals for the Cartesian form of complex numbers? What might be a possible learning goal for a Cartesian form of complex numbers as a mathematical concept? What might be a possible sequence of tasks affording the learning goal through one's own activity for a Cartesian form of complex numbers?

Learning Through Activity and Complex Numbers

Learning Through Activity instructional design is composed of four steps: The first two steps for the generation of HLT are determining students' current knowledge and identifying a learning goal. The third step is specifying "…an activity that students have currently available that can be the basis for the abstraction specified in the learning goal… The fourth step is the design of the task sequence" (Simon et al, 2018, p. 104). In particular, the trajectory for complex numbers is based on the assumption that, at the outset, the students currently can define any quadratic function with real coefficients and graph them on the Cartesian coordinate systems and know that given any quadratic equation, $ax^2 + bx + c = 0$ where $a, b, c \epsilon R$, $a \neq 0$ the two roots of the quadratic equation are of the form $x_{1,2} = \frac{-b}{2a} \pm \frac{\sqrt{\Delta}}{2a}$, where $\Delta = b^2 - 4ac$ and is called the discriminant (Δ). They also know that x_1 and x_2 can be located on the Real number line, as $(\frac{-b}{2a} - \frac{\sqrt{\Delta}}{2a}, 0)$ and $(\frac{-b}{2a} + \frac{\sqrt{\Delta}}{2a}, 0)$ respectively. Here, the component $\frac{-b}{2a}$ indicates $(\frac{-b}{2a}, 0)$, the abscissa-of-the-vertex (as well as the symmetry axis) and has a distance to the origin and a distance to the roots, that is $\frac{\sqrt{\Delta}}{2a}$ (Hedden & Langbauer, 2003). Regarding the learning goal, we first elaborate on the following: Mathematically, any complex number $z = x + iy$ is an ordered pair (x,y) of real numbers x and y, with $i = \sqrt{-1}$. However, mathematical definitions or theorems are not necessarily the same as mathematical concepts (Vergnaud,1997). Simon (2017) defined a mathematical concept as " a researcher's articulation of intended or inferred student knowledge of the logical necessity involved in a particular mathematical relationship" (p. 123). So, we concur that a researcher's articulation of such a learning goal might be as follows: "Complex numbers, located in the complex plane as the solutions of a quadratic equation f(x)=0 with a fixed apsis of the vertex, vary continuously with the coefficients of f(x) such that the continuous changes in directed distances of the solutions to the apsis of the vertex and to the origin result in the changes in the locations of the solutions in the complex plane". In the LTA design, the third step starts with asking the question, "What activity, currently available to the students, might be the basis for the intended learning?" (Simon & Tzur, 2004, p. 96). Thus, for the learning goal, we identify the specified activity as continuously varying the locations in the complex plane of the roots of any quadratic equation with the same apsis of the vertex. For the fourth step in LTA, Simon et al. (2018) stated, "The task sequence must both elicit the intended student activity and lead to the eventual coordination of actions on the part of the students" (p.104). Thus, we generated the following tasks: *The first set of tasks: 1)* How many parabolas are there with the same abscissa of the vertex? 2)Draw what you imagine in the first question. 3) What changes and remains invariant in the parabolas you have drawn given the algebraic form, $f(x) = ax^2 + bx + c$? Why? 3a) How are the distances of the roots to the-apsis-of-the-vertex changing or not changing? *The second set of tasks*: 1) Given a set of quadratic functions such as $x^2 + 2x$ -8, $x^2 + 2x$ -4, $x^2 + 2x$ -1, $x^2 + 2x$, and $x^2 + 2x$+1 and their graphs as parabolas on a) Desmos and b) colored copy, what changes and what remains invariant? Why? 2) How are the distances of the roots and the abscissa-of-the-vertex changing or not changing? 3) Locate the roots on the Real number line from #1. How do the changes in the distances of the roots to the abscissa of the vertex relate to the different forms of Δ? 3a) How do you relate the changes in the distances of the roots to the abscissa of the vertex when the increment is bigger than zero, is zero, and is smaller than zero? 4) How could you re-write and locate the roots when the value of $\Delta = b^2 - 4ac$ becomes negative?

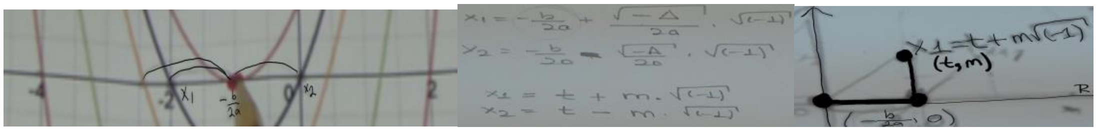
Methods

We report results from a teaching experiment (Steffe and Thompson, 2000). The first author acted as the teacher-researcher in the teaching experiment, which consisted of three 75- to 120-minute sessions. Data sources included transcripts formed from the video data and written artifacts. The study's participant was one prospective secondary mathematics teacher, Esra, who was in the fourth year of her five-year undergraduate program. Following the teaching experiment methodology, analysis was both ongoing and retrospective (Steffe & Thompson, 2000). Ongoing analysis occurred during data collection and involved formulating hypotheses of student thinking between sessions and designing subsequent sessions to test those hypotheses. For the retrospective data analysis, we followed three-level approach to abductive process (Simon, 2019). For the first level, we read the transcripts line-by-line, focusing on sequences in which Esra's actions and utterances provided information about her thinking. Then, "*we use the results of the first level as the "data" for the second level*, making inferences for chunks of these new data" (Simon, 2019, p. 119) by answering the questions such as what understandings Esra seems to have and how we can characterize her thinking. The third level involved our use of explanatory constructs, such as learning as a reflection on the activity.

Results

In lieu of space, we describe Esra's evolvement of ideas on the second set of tasks. For the second set of tasks, 1 and 2, given the examples on Desmos, Esra stated that the values of 'a' and 'b' and so, $\frac{-b}{2a}$, remained invariant in the examples and the values of 'c' changed, decreasing the roots' distances to the abscissa of the vertex (from out to inwards). She also commented on the roots, stating, "…with respect to the symmetry axis, they [*the different roots*] have the same, umm, they [*the roots*] are symmetric, exactly". Also, she further commented on the roots' distances to the abscissa of the vertex as changing. For 3 and 3a in the second set of tasks, Esra drew the following (*See* Figure 1). She did not use the exact values of the roots, but she marked some of the roots on the real number line as (x_1,0), (x_2,0), ($x_{1'}$,0), ($x_{2'}$,0), ($x_{1''}$,0), ($x_{2''}$,0), ($x_{1,2'''}$,0).

Figure 1. Esra's showing the roots on real number line she drew



a *b* *c*

Figure 2. Esra's pointing to the real numbers as possible roots, re-writing the roots of the quadratic equations and plotting the roots of the quadratic equation on the plane

Furthermore, Esra re-wrote the roots for 4 in the second set of tasks (*See* Figure 2b) and stated: "Let's say there are infinitely many [*quadratic*] functions, and the abscissa-of-the-vertex of any quadratic function is t...and m is the distance from one of the roots to the abscissa-of-its-vertex…" Then, albeit with struggle, she positioned the roots on the plane(*See* Figure 2c) and commented on what (t,m) represented:

E: Umm, the root of the equation. The roots. They are the roots of the equation, quadratic equation. Umm, when they are not real, the delta is smaller than 0. Umm, when they are real, delta is greater than 0.

R: What do you call the algebraic expressions when delta is smaller than 0?

E: Umm complex numbers.

R: Okay. Where do you get those complex numbers?

E: From real numbers. All real numbers, on the real x-axis…Umm, I obtain them from the real roots of quadratic equations. If they are umm...okay, I obtain them from their real roots. Okay, I obtain from non-real ones as well […] Yes, they are complex numbers. The numbers obtained from the roots of all quadratic equations are complex numbers. Exactly. They give complex numbers.

Importantly, the data showed that Esra enacted the activity several times, such as in the first set of tasks, on the examples provided on Desmos and the colored-print copy (*See* Figure 2a), and when placing them as points on the real number. This way, her repeated mental runs through the activity of varying continuously the roots' locations on first the real number line and then on the plane allowed her images to become operative such that reflecting on the activity with the three cases of discriminant allowed her to anticipate that all the roots constituted the elements of a set of the roots of quadratic equations with real coefficients.

Conclusion and Discussion

We presented learning through activity design principles focusing on an advanced mathematical topic, the Cartesian form of complex numbers. Tasks designed with the principles of the LTA framework are distinguished from other task designs (e.g., Smith & Stein, 1998) in explaining the relationship to students' learning processes. Data indicated that engaging in the task sequence, Esra's enactment of the activity of varying continuously the locations of the roots of any quadratic equation allowed her to anticipate the logical necessity that any quadratic equation having both two real and two non-real number roots constitutes the elements of the set of complex numbers. Results suggest that the LTA design principles might be used to study advance mathematical concepts. We also argue that the learning goal shared in this paper might

Kosko, K. W., Caniglia, J., Courtney, S., Zolfaghari, M., & Morris, G. A., (2024). *Proceedings of the forty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Kent State University.

constitute a considerable change in students' conception of complex numbers. Previous research suggests that learners from different stages of schooling seem to struggle with the idea that any real number is a complex number (Nordlander &Nordlander, 2012). Thus, we argue that the aforementioned learning goal with the design of the tasks might allow to conceptualize real numbers as a subset of complex numbers.

Acknowledgments

Opinions and conclusions in this paper are those of the authors. Data in this paper came from Merve Saraç's Master Thesis as part of a larger project conducted under the supervision of Gülseren Karagöz Akar. This paper is supported by The Scientific and Technological Research Council of Türkiye (TUBITAK, Grant No: 2219).

References

Common Core State Standards Initiative. (2010). *Common Core State Standards for Mathematics (CCSSM)*. National Governors Association Center for Best Practices and the Council of Chief State School Officers. https://learning.ccsso.org/wp-content/uploads/2022/11/ADA-Compliant-Math-Standards.pdf

Dreyfus, T. (2018). Learning Through Activity-Basic research on mathematics cognition. *Journal of Mathematical Behavior, 52,* 216–223. https://doi.org/10.1016/j.jmathb.2018.04.001

Gravemeijer, K. (2004). Learning trajectories and local Instruction theories as means of support for teachers in reform mathematics education. *Mathematical Thinking and Learning, 6*(2), 105–128.

Hedden, C.B., & Langbauer, D. (2003). Balancing problem-solving skills with symbolicmanipulation skills, In H.L. Schoen, R.I. Charles (Eds.). *Teaching mathematics through problem solving: Grades 6-12* (pp. 155–159). National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Nordlander, M.C., & Nordlander, E. (2012). On the concept image of complex numbers. *International Journal of Mathematical Education in Science and Technology, 43*(5), 627–641. https://doi.org/10.1080/0020739X.2011.633629

Smith, M. S., & Stein, M. K. (1998). Selecting and creating mathematical tasks: From research to prac- tice. *Mathematics Teaching in the Middle School, 3,* 344–50.

Simon, M.A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26,* 114–45. https://doi.org/10.2307/749205

Simon, M., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning*, *6*, 91–104. https://doi.org/10.1207/s15327833mtl0602_2

Simon, M.A., Saldanha, L., McClintock, E., Karagoz Akar, G., Watanabe, T., & Zembat, I.O. (2010). A developing approach to studying students' learning through their activity, *Cognition and Instruction*, *28*(1), 70–112. https://doi.org/10.1080/07370000903430566

Simon, M.A. (2017). Explicating mathematical concept and mathematical conceptionas theoretical constructs. *Educational Studies in Mathematics, 94*(2), 117–137. https://doi.org/10.1007/s10649-016-9728-1

Simon, M.A., Kara, M., Placa, N., & Aviztur, A. (2018). Towards an integrated theory of mathematics conceptual learning and instructional design. The Learning Through Activity theoretical framework, *Journal of Mathematical Behavior, 52*, 95–112. https://doi.org/10.1007/s10649-016-9728-1

Simon, M.A. (2019) Analyzing qualitative data in mathematics education. In: Leatham K. (Eds.), *Designing, conducting, and publishing quality research in mathematics education (pp. 111-122).* Springer.
Singapore higher 2 maths curriculum. (n.d.). Retrieved February 10, 2024 from https://sg.ixl.com/standards/maths/higher-2

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.). *Research design in mathematics and science education* (pp. 267–307). Hillside, NJ: Erlbaum.

Tzur, R. (2018). Simon's team's contributions to scientific progress in mathematics education: A commentary on the Learning Through Activity (LTA) research program. *Journal of Mathematical Behavior, 52,* 208–215. https://doi.org/10.1016/j.jmathb.2018.02.005

Vernagud, G. (1997). The nature of mathematical concepts. In T.Nunes & P.Bryant (Eds.). *Learning and teaching mathematics: An international perspective* (pp. 5–28). East Sussex UK: Psychology Press.