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OPERATIONALIZING RE-PRESENTATION TO INVESTIGATE AND SUPPORT STUDENTS' COVARIATIONAL REASONING

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Within the body of work on students' covariational reasoning, researchers have called for more explicit attention to the ways theoretical constructs are operationalized to develop characterizations of student thinking. Addressing this need, we outline how von Glasersfeld's (1991) notion of re-presentation—the act of reconstructing something previously experienced in its absence—has informed our research program on students' covariational reasoning. Specifically, we illustrate its multimodal use in framing claims regarding the extent a student has constructed a particular covariational relationship.

INTRODUCTION

Covariational reasoning refers to the mental operations involved in coordinating two quantities' magnitudes or values as they vary in tandem (Thompson & Carlson, 2017). Students' covariational reasoning remains a growing area of study due to researchers having illustrated its critical foundation for students constructing major algebra, function, calculus, and STEM concepts (Thompson & Carlson, 2017). Accordingly, researchers have provided a variety of models of student thinking, with each model entailing the use of theoretical constructs to make aspects of student thinking salient. For instance, Carlson et al. (2002) specified several mental actions associated with students' covariational reasoning. Similarly, Ellis et al. (2020) and Johnson (2015) have each characterized nuances in the ways students reason about covariation.

A by-product of growth in an area of study is that guiding theories and constructs become more or less noticeable as researchers develop more nuanced or detailed characterizations. For example, as researchers have developed more specified descriptions of the mental actions involved in students' covariational reasoning, macro-level constructs that focus on general properties or forms of reasoning have moved to the background. This progression is natural and often necessary, yet it has notable consequences (Tyburski et al., 2021). For one, it leaves unclear the ways in which macro-level constructs emerged and continue to inform research design or analysis. For another, it inhibits other researchers adopting the work for their own purposes. In a call to fellow researchers, Tyburski et al. (2021) argued these consequences negatively impact the accessibility of research to novice or outsider researchers.

We respond to this call by identifying the ways von Glasersfeld's (1991) notion of re-presentation—the act of reconstructing something previously experienced in its absence—has informed our research program on students' covariational reasoning. In

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what follows, we first provide background information on re-presentation and students' covariational reasoning. We then discuss the explicit ways in which re-presentation has emerged and informed our research. Namely, we have used re-presentation to consider and frame the viability of our claims regarding students' covariational reasoning.

RE-PRESENTATION

von Glasersfeld's notion of re-presentation emerged during his study of Piaget's genetic epistemology and as a distinction relevant to object permanence (von Glasersfeld, 1991, 1995). Re-presentation refers to the ability of an individual to construct a visualized image of an object in the absence of the relevant sensory material. von Glasersfeld emphasized the hyphenated form of re-presentation for two primary reasons. As the first reason, the hyphenated form reflects that to both von Glasersfeld and Piaget, re-presentation is an *active* attempt to present again. Because re-presentation involves regenerating a past experience or concept in the absence of the relevant figurative material, it is subject to and defined by the ways of operating available to the individual at that moment. Re-presentation does not produce a copy of the previous experience or concept, nor is it a simple recall of the previous experience as with a ready-made picture. Relatedly, because re-presentation is an active process, a researcher should not presume that the operations involved in re-presentation are equivalent to those used during the initial experience. This is particularly true when a large duration of time separates the two. As the second reason, von Glasersfeld's insistence on using the hyphenated re-presentation reflects his linguistics background. He desired to distinguish between *re-presentation* and *representation*. Whereas the former is a constructive process involving the enactment of conceptual structures, he defined the latter as something acting as a copy, a pointer, or something that stands in for something else (von Glasersfeld, 1995). For instance, one might say a displayed Cartesian line and the inscription "$y = 3x$" represent (without hyphen) a linear relationship, whereas a re-presentation (with hyphen) of a linear relationship involves enacting conceptual operations associated with the conceived relationship to regenerate associated figurative material. We expand on this example in the next section.

Further emphasizing its importance for the construction of concepts, von Glasersfeld described re-presentation as one of the key drivers of abstraction and learning. He considered the re-presentation of objects and conceptual structures to enable the construction of hypothetical situations not available on an experiential or sensorimotor basis. In his words, re-presentation enables thought experiments, and through affording processes of abstraction "thought experiments constitute what is perhaps the most powerful learning procedure in the cognitive domain" (von Glasersfeld, 1995, p. 69). As an apropos example, Steffe and colleagues' (Steffe & Olive, 2010) extensive research program on fractional reasoning illustrates that acts of re-presentation are inseparable from the construction of number and multiplicative reasoning.

MAGNITUDES, OPERATIONS, AND COVARIATION

Research on covariational reasoning, or "reasoning about values of two or more quantities varying simultaneously" (Thompson & Carlson, 2017, p. 423), has primarily occurred within Thompson's quantitative reasoning paradigm. Informed by von Glasersfeld's radical constructivism and Piaget's genetic epistemology, Thompson defined a *quantity* as a measurable attribute of some situation (Thompson, 1989). Reflecting the theory's epistemological underpinning, Thompson emphasized that quantities and their relationships are cognitive constructions and thus idiosyncratic to the knower. Researchers have since adopted this perspective to develop insights into the quantities and covariational relationships students and teachers construct (see Thompson & Carlson, 2017 for a summary of this work). We focus on two aspects from this work in order to connect re-presentation to students' covariational reasoning.

Firstly, a fundamental distinction in Thompson's theory is that between quantitative operations and arithmetic operations (Thompson, 1989). The former refers to the mental operations involved in constructing a quantity and associated amountness, while the latter refers to numerical operations that define or calculate a quantity's measure or value. To clarify, consider using the inscriptions "2" or "6-4" to represent a measure or comparison between measures. Here, represent (no hyphen) is used in the sense of their standing in for or pointing to anticipated conceptual (quantitative) structures. Because re-presentation stresses the *enactment* of mental operations in order to reconstruct a conceptual structure (von Glasersfeld, 1995), re-presenting "2" involves reconstructing quantitative operations including creating and iterating a unit magnitude in the context of figurative material that permits those operations (e.g., a segment). With respect to the inscription "6-4", an act of re-presentation involves reconstructing those same operations for "6" and "4", and then reconstructing the operations involved in disembedding and measuring the magnitude by which the "6" length exceeds the "4" length (Thompson, 1989). Underscoring the difference between re-presenting operations and representing, we suspect the reader immediately understands "2" as representing the result of evaluating "6-4" without having to enact in re-presentation the operations represented by "6-4" or the additive difference of "2".

Secondly, Carlson et al. (2002) provided a framework of mental actions that specify several quantitative operations involved in covariational reasoning. For the purposes of this paper, we draw attention to *direction of change* and *amount of change* operations. Direction of change involves conceiving variation in one quantity's magnitude in tandem with variation in another quantity's magnitude. For instance, in the context of counter-clockwise circular motion from a 3 o'clock position, the height above the circle's center increases as the arc length traversed increases (Figure 1). Here, the quantities' magnitudes are paired while each quantity's magnitude is compared across states via a gross comparison with its previous state. Amount of change involves further quantifying quantities' covariation by systematically comparing the accumulation of each quantity. As an example, one can capture the arc length's accumulation by constructing and iterating a unit arc length. Pairing height with the arc length's accumulation, the individual can construct and additively compare not only successive heights, but also the successive variations in height (Figure 1). Here, the variations in both quantities' magnitudes are coordinated, with one quantity's variation remaining equivalent in magnitude (i.e., equal, successive increases) while the variation in the other quantity's magnitude is compared across states via a gross comparison (i.e., the increase is decreasing). We underscore that this illustration centers quantitative operations, magnitudes, and associated figurative material, as opposed to specified values, inscriptions representing those values, or arithmetic operations involving values. Each are critical for mathematical development and communication, but acts of re-presentation involve the reconstruction of the former.

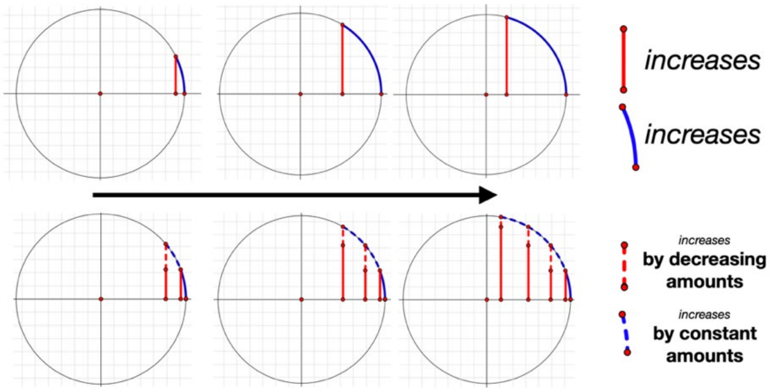

Figure 1: Direction of change (top) and amount of change (bottom).

RESEARCH CONTEXTS

This paper emerged from the empirical work of building accounts of student thinking in the context of major algebra, pre-calculus, and calculus ideas. The primary attention of this work has been understanding, engendering, and supporting students' and teachers' quantitative and covariational reasoning. The work involved a series of teaching experiments with middle-grade, secondary, and undergraduate students and teachers. A *teaching experiment* is a qualitative design-based research methodology that involves constructing and testing hypothetical models of student thinking (Steffe & Thompson, 2000). Analytic methods of conceptual analysis (Steffe & Thompson, 2000) in combination with generative and convergent coding (Corbin & Strauss, 2008) accompanied the teaching experiments. It was during the iterative execution and analyses of the teaching experiments that re-presentation emerged as a useful construct, and we point the reader to Stevens (2019), Liang and Moore (2021), and Moore et al. (2022) for specified accounts of and references to this empirical work and findings.

RE-PRESENTATION AND CLAIM VIABILITY

The initial need for re-presentation as an explanatory construct emerged when our research team noticed a similar phenomenon during a series of studies: a student had engaged in activity that strongly suggested their having constructed a stable understanding of some covariational relationship, but their actions during subsequent tasks suggested otherwise. For example, in exploring circular motion, we experienced students repeatedly producing diagrams consistent with Figure 1 along with the appropriate verbal descriptions. The fluidity of their actions led us to believe they had constructed a sophisticated and stable covariational relationship. However, the student would experience difficulties when prompted to construct a Cartesian graph of the relationship, or to choose two segments that match the covariational relationship from a collection of varying segments. The difficulties occurred in two primary ways.

In some cases, a student's difficulty would occur when they attempted to return to and regenerate the original situation and relationship in the presence of a new task. As an example, a student named Lilly attempted to regenerate the relationship illustrated in Figure 1 when attempting to determine which two segments from a collection of varying segments captured the sine relationship (Figure 2). Illustrated in detail in Liang and Moore (2021), Lilly desired to use the displayed circle to regenerate the relationship she previously determined as "sine" so she could compare it with how chosen segment-pairs covaried. However, she experienced difficulty regenerating the relationship unless the researchers provided figurative material (e.g., marks to visually denote amounts of change) to support her in making quantitative comparisons.

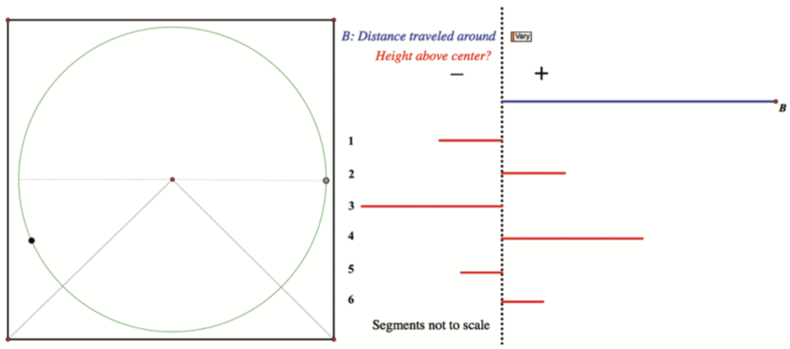

Figure 2: Choosing from six (red) varying segments (Liang & Moore, 2021, p. 300).

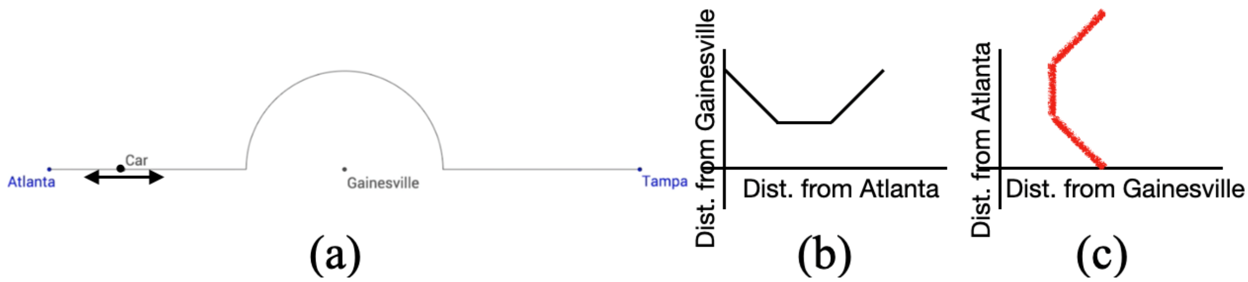

Figure 3: The (a) task situation and (b-c) normative graphs.

In other cases, a student would return to and regenerate the original situation and relationship without trouble, but the student would experience a difficulty regenerating a previously constructed relationship using the figurative material of a new task. As an example, after determining a covariational relationship in a situation and constructing a graph of that relationship by re-presenting the quantities' covariation (Figure 3a-b), Moore et al. (2019) reported on a student abandoning the construction of the graph in an alternative Cartesian coordinate orientation (i.e., the axes swapped, Figure 3c). The student, Patty, experienced no issues regenerating the covariational relationship in the situation or using the initial coordinate orientation, but she perceived creating a graph in the new coordinate orientation as requiring drawing it “right-to-left.” She claimed such a graph is “backwards” and must be incorrect because of that feature.

The frequency of cases like these in tandem with the students’ experienced difficulties being sustained and significant led us to question the extent we could claim the students’ reasoning foregrounded covariational reasoning. In Lilly’s case, we perceived her difficulties in re-presenting the relationship in her previously experienced context to be a contraindication of such reasoning. In Patty’s case, her difficulties in re-presenting the relationship under a new coordination orientation were also a contraindication of such reasoning. We thus searched for a construct that could help us not only characterize each case, but also differentiate between them.

We do not recall the first instance in which we came across re-presentation as a potential tool. But, it became clear that re-presentation would be a useful tool when a research team member was in the depths of her dissertation work and needed to distinguish between students’ uses of formulas as inscriptions capturing arithmetic rules between values or as symbolizing quantitative operations relevant to a dynamic geometric object (Stevens, 2019). Upon coming across re-presentation, our team returned to our data to engage in further rounds of conceptual analysis. In doing so, re-presentation’s dual emphasis on the availability of figurative material and the reenactment of conceptual operations provided us a way to situate our claims regarding students’ reasoning so that we considered them viable. Here, our use of viable is compatible with Steffe and Thompson (2000). We consider a claim viable if it is both an adequate hypothetical account of student thinking and it is specified enough to convey both affordances and constraints in their reasoning.

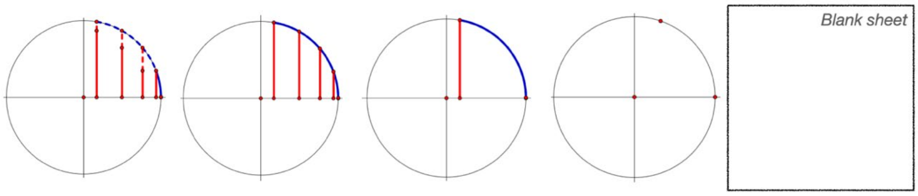

Figure 4: Varying the provided figurative material.

Reflecting on the cases above and considering the dual emphasis of re-presentation, we can explore indications and contraindications regarding students’ covariational reasoning in two ways after a student has engaged in activity that we take as providing evidence of covariational reasoning. Firstly, as researchers, we can prompt a student to re-present their actions within the same context or phenomenon as previously experienced. Furthermore, we can vary the amount of figurative material provided to them. For instance, after a researcher has evidence a student has constructed the relationship consistent with Figure 1, during a subsequent task the researcher could prompt the student to reconstruct that relationship, and they could do so in a way that provides a range from a completed diagram to a blank sheet of paper (Figure 4). Returning to Lilly, when only provided a dynamic point on a circle, she could not re-present her previously constructed relationship. But, when provided the collection of heights all at once, she was able to re-present her previously constructed relationship.

Secondly, we can prompt students to re-present their actions within a different (or series of different) context(s) or phenomenon(s). For instance, after a researcher has evidence a student has constructed a relationship in a phenomenon (e.g., circular motion or a road trip), the researcher could prompt the student to reconstruct that relationship within a variety of Cartesian orientations (e.g., Figure 3), alternative coordinate systems (e.g., polar coordinates), or number line situations (e.g., Figure 2). A researcher can also vary the amount of figurative material available in the new contexts or phenomenon. For instance, in moving to a different coordinate system (e.g., polar coordinates), a researcher may or may not provide a quantity's variation partitioned (e.g., a marked grid). Such moves support a researcher in differentiating between a student's understanding of a particular covariational relationship and their generalized understanding of the coordinate system's quantitative structure. Returning to students like Patty, if a student considers drawing a graph "left-to-right" to be absolutely necessary, then no amount of figurative material would immediately support them in drawing and accepting a normative graph in the given orientation. On the other hand, in the original Cartesian orientation, Patty was able to re-present her relationship. Illustrating how re-presentation supports a researcher in situating their claims, Patty's actions indicate that she had constructed a covariational relationship she could re-present graphically, but her Cartesian graphing meanings entailed properties of movement that did not support her in doing so for a particular orientation.

CLOSING COMMENTS

von Glasersfeld's notion of re-presentation enables a researcher to situate their claims regarding a student's covariational relationship with respect to 1) the amount of figurative material necessary to re-present the relationship, 2) their ability to re-present the relationship in other contexts and phenomenon, and 3) a combination of the two. By designing task environments sensitive to these re-presentational framings, we can systematically pursue indications and contraindications of students having constructed particular covariational relationships based on their capacity to re-present those relationships. Importantly, adopting a re-presentation framing has increased our sensitivity to the properties and features that students abstract from their their initial construction of a quantitative or covariational relationship. This supported sensitivity underscores von Glasersfeld's framing of re-presentation as a driver of learning.

On the topic of learning, it is important to note that students' re-presentational activity can and does change over time. What a student is able to re-present from one day to another might not be available to them at a later time. Likewise, what a student cannot re-present at one moment in time may become available to them in re-presentation at another moment in time. This phenomenon is inherent to the learning process and cognitive development (Steffe & Olive, 2010; von Glasersfeld, 1995), and it suggests that instruction and curricular materials should give direct attention to engendering and supporting cycles of students' re-presentational activity. Not only is re-presentation a driver of abstraction, it is a precursor to meaningful symbolization, and thus it provides a foundational springboard to an individual's mathematical development. By responding to the call by Tyburski et al. (2021), we hope to not only provide insights into how re-presentation has emerged in our research, but also invite conversation about how it might inform the teaching and learning of mathematics more broadly.

References

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352-378.

Corbin, J. M., & Strauss, A. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (3rd ed.). Sage.

Ellis, A. B., Ely, R., Singleton, B., & Tasova, H. (2020). Scaling-continuous variation: supporting students' algebraic reasoning. *Educational Studies in Mathematics*.

Johnson, H. L. (2015). Secondary students' quantification of ratio and rate: A framework for reasoning about change in covarying quantities. *Mathematical Thinking & Learning*, *17*(1), 64-90.

Liang, B., & Moore, K. C. (2021). Figurative and operative partitioning activity: A student's meanings for amounts of change in covarying quantities. *Mathematical Thinking & Learning*, *23*(4), 291-317.

Moore, K. C., Liang, B., Stevens, I. E., Tasova, H. I., & Paoletti, T. (2022). Abstracted Quantitative Structures: Using Quantitative Reasoning to Define Concept Construction. In G. Karagöz Akar, İ. Ö. Zembat, S. Arslan, & P. W. Thompson (Eds.), *Quantitative Reasoning in Mathematics and Science Education* (pp. 35-69). Springer International.

Moore, K. C., Stevens, I., Paoletti, T., Hobson, N., & Liang, B. (2019). Pre-service teachers' figurative and operative graphing actions. *The Journal of Mathematical Behavior*, *56*.

Steffe, L. P., & Olive, J. (2010). *Children's Fractional Knowledge*. Springer.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-307). Erlbaum.

Stevens, I. E. (2019). *Pre-service teachers' constructions of formulas through covariational reasoning with dynamic objects* [Ph.D. Dissertation]. University of Georgia: USA.

Thompson, P. W. (1989, March 27-31). *A cognitive model of quantity-based algebraic reasoning* Annual Meeting of the AERA, San Francisco, CA.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 421-456). National Council of Teachers of Mathematics.

Tyburski, B., Drimalla, J., Byerley, C., Boyce, S., Grabhorn, J., & Moore, K. C. (2021). From theory to methodology: Guidance for analyzing students' covariational reasoning. In D. Olanoff, K. Johnson, & S. M. Spitzer (Eds.), *Proceedings of the 43 annual meeting of the PME-NA*.

von Glasersfeld, E. (1991). Abstraction, re-presentation, and reflection: An interpretation of experience and of Piaget's approach. In L. P. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 45-67). Springer.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Falmer Press.