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EXPANDING SLOPE UNDERSTANDING: THE COMPOSED UNIT RATIO

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This study investigates slope as a composed unit ratio, offering an alternative understanding that diverges from a multiplicative comparison meaning. Through the lens of a teacher's unique interpretation, we bridge the gap between the different meanings of slope and the varied understandings of ratio, uncovering a nuanced meaning of slope as a ratio. Our findings suggest that viewing slope as a composed unit ratio offers accessible and meaningful pathways for learners, by highlighting a productive understanding. We advocate for further exploration into this meaning to enrich pedagogical strategies and support the development of robust mathematical understandings.

Keywords: Algebra and Algebraic Thinking, Teacher Knowledge

The slope concept is foundational for mathematical learning. Traditionally, slope has been understood and taught through several lenses, including rise over run, algebraic formulas, ratios, and as a measure of line steepness. These conceptions, as outlined by Nagle and Moore-Russo (2013) and other researchers (e.g., Byerley & Thompson, 2017; López et al., 2024), are pervasive in educational research and practice, offering valuable insights into the different ways teachers conceptualize slope. However, these frameworks often consider slope-as-ratio to mean a multiplicative comparison—an understanding recognized for its depth but noted for its scarcity among both teachers and students (Cho & Nagle, 2017; DeJarnette et al., 2020). In this study, we propose that understanding slope as a composed unit ratio may provide more accessible and equally rigorous pathways to deep mathematics.

Despite extensive analysis of the challenges surrounding the teaching and learning of slope, the exploration of composed unit reasoning as a conceptual foundation for slope has been relatively underexamined. Our research addresses this gap through the lens of a teacher, and specifically her interpretation of slope as a composed unit ratio. By investigating the intersection of the bodies of research regarding the different meanings of slope and the various interpretations of ratio, we illustrate that the teacher's conceptualizations of slope, while not aligning with a multiplicative comparison meaning, encompass other rich, nuanced understandings of ratio. The introduction of this alternative meaning of slope not only broadens the conceptual repertoire available for teaching slope but also has the potential to support the development of productive mathematical meanings among learners. Thus, this manuscript seeks to answer the question: What characterizes a ratio-as-composed-unit meaning for slope, and what are the affordances and constraints of this meaning?

Background and Theoretical Framework: Teachers' Meanings of Slope

Researchers have characterized a number of different meanings teachers hold for slope, many building from Nagle and Moore-Russo's (2013) 11 conceptualizations of slope. These meanings include slope as a geometric ratio, an algebraic ratio, and a physical property, among others. Synthesizing the literature base as a whole (e.g., Byerley & Thompson, 2017; Coe, 2007; López et al., 2024; Stump, 1999), three of the most prevalent teacher meanings for slope are (a) slope as an index of the steepness of a line, (b) slope as rise over run, and (c) slope as ratio. We discuss each of these in turn.

Slope-as-steepness meaning entails conceiving of a line as a physical object and making perceptual associations between its steepness and a numerical value; Nagle and Moore-Russo (2013) called this conception "physical property" (p. 3). When holding this conception, one might conclude that the slopes of two lines are the same if they have the same steepness (as determined visually), even if they are graphed in coordinate systems with different scales. This is a fairly common conception for both pre-service teachers (Avcu & Biber, 2022; Paulucci & Strepp, 2021; Tasova & Moore, 2018) and in-service teachers (Byerley & Thompson, 2017; López et al., 2024; Stump, 1999). For instance, Tasova and Moore (2018) found that one pre-service teacher's meaning of slope as a measure of steepness hindered her ability to recognize consistency across graphs in different coordinate orientations.

Slope-as-rise-over-run meaning entails thinking about slope as a procedure for moving up and over a specified number of units on a Cartesian coordinate plane, what Nagle and Moore-Russo (2013) called "geometric ratio" (p. 3), or when determined by the slope formula, $\frac{y_2-y_1}{x_2-x_1}$, "algebraic ratio" (p. 3). In Nagle and Moore-Russo's study, these were two of the most common conceptions. Multiple researchers have found that most teachers' meanings for slope include the slope formula (e.g., Byerley & Thompson, 2017; López et al., 2024; Stump, 1999; 2001). Although teachers can articulate the slope formula as a ratio, Byerley and Thompson (2017) showed that for many teachers this conception is non-multiplicative, as there is no attention to the change in one quantity compared to the change in the other quantity.

Slope-as-ratio meaning is a consequence of comparing the changes in two quantities multiplicatively to create an emergent quantity (Ellis, 2007). This entails understanding slope as a measure of one quantity's variation with respect to the variation of another quantity. For instance, one can conceive of speed as an emergent quantity through the multiplicative comparison of change in distance to change in time (Sherin, 2000). Few teachers refer to slope in this manner, and they can struggle to explain the use of division in the slope formula (e.g., Avcu & Biber, 2022; Byerley & Thompson, 2017; Coe, 2007; Talib et al., 2023). At the same time, slope-as-ratio meaning "is particularly powerful in that it supports one's ability to make sense of slope in a variety of situations" (Diamond, 2020, p. 166). A slope-as-ratio meaning is taken as evidence of a deeper understanding of slope and is critical for making connections between a slope value and the constant rate of change in a linear function (DeJarnette et al., 2020; Dolores Flores et al., 2020; Lobato & Siebert, 2002; Talib et al., 2023). Understanding slope as a ratio supports the ability to conceptualize the invariability of slope (Deniz & Kabael, 2017), to transfer the slope concept to other contexts (Hoban et al., 2013), and to use algebraic manipulations to determine slopes effectively (Cho & Nagle, 2017).

Two Ways to Understand Slope as a Ratio

Research addressing slope as a ratio typically considers ratio to mean a multiplicative comparison (Hoban, 2021; Talib et al., 2023). However, it is also possible to think of a ratio as a *composed unit*. A composed unit is created by joining two quantities to create a new unit, such as 5 cm: 2 sec (Lamon, 1994; Lobato & Ellis, 2010). One can then iterate or partition the created composed unit, maintaining the simultaneity between those quantities while creating new units (Jacobson et al., 2018). For instance, a person developing speed as a ratio can think of an inchworm crawling 5 centimeters in 2 seconds. The composed 5 cm : 2 sec unit could then be iterated to create other equivalent ratios, such as 10 cm : 4 sec, 20 cm: 8 sec, and so forth. This unit can also be partitioned to create, for instance, a unit ratio of 2.5 cm: 1 sec or 1 cm : 2/5 sec. Although some researchers consider the composed unit to be pre-ratio reasoning (e.g., Lesh et al., 1988), others point out that it can be used in combination with other concepts to develop a robust understanding of proportionality (Ellis, 2013; Lobato & Ellis, 2010).

In contrast, a *multiplicative comparison* entails considering how many times larger one quantity is compared to the other (Kaput & Maxwell-West, 1994; Lobato & Ellis, 2010). To continue the above example, this means understanding that the inchworm travels 2.5 cm for every second, or the number of centimeters traveled is always 2.5 times as large as the number of seconds. Regardless of whether one creates a composed unit or makes a multiplicative comparison, both ways of reasoning entail keeping the ratio of one quantity invariant to the other as the numerical values of both quantities change by the same factor (Aydeniz Temizer, 2022).

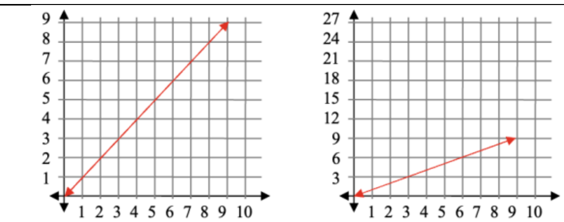
We found only one study, by DeJarnette and colleagues (2020), that distinguished between ratio as composite unit and ratio as multiplicative comparison in relation to slope meanings. The authors claimed that interpreting ratio as a multiplicative comparison, which they called a single value, is necessary for a sophisticated understanding of slope. However, we suspect that limiting the slope-as-ratio meaning strictly to multiplicative comparisons might miss instances in which teachers (or students) are beginning to build a multiplicative understanding by iterating and partitioning composed units. Given the documented difficulties teachers have with reasoning with slope as a multiplicative comparison, it may be fruitful to consider instances in which teachers are understanding slope as composed units and reasoning with such units in order to build notions of invariance. In our study, we present a case of a teacher whose interpretation of slope as a ratio of composed unit showcases a quantitative and productive understanding, thereby enabling a meaningful engagement with various scenarios.

Methods

This study is part of a larger investigation aimed at understanding how teachers support mathematical generalizing (e.g., Ellis et al., 2024). Within this broader project, we identified Ms. R, a sixth-year high school algebra teacher, for an in-depth case study due to her insights into the teaching and understanding of slope. We adopted an investigative and descriptive case study approach (Merriam, 1998; Yin, 2009) to explore Ms. R's meanings of slope. This paper reports on findings from three semi-structured clinical interviews (Ginsburg, 1997) with Ms. R designed to probe her conceptualizations of and MKT related to slope. The 90-minute interviews focused on her understandings of slope and ratio, her insights into how students develop these understandings, and her strategies for supporting its development. This paper concentrates on Ms. R's personal meanings of slope, and our analysis of her broader $\text{MKT}_{\text{slope}}$ is reported elsewhere (Tasova et al., 2024).

To investigate Ms. R's meanings of slope, we employed a set of questions inspired by Diamond (2020), such as Ms. R's spontaneous associations with the term "slope," contexts or situations she associates with slope, and her interpretation of specific slope values such as ½ and −1. Further, we introduced Ms. R to tasks, such as the "Five Students Problem" (adapted from Diamond, 2013; see Figure 1a), designed to reveal the extent to which her various meanings of slope. We asked Ms. R about what she would say or do with each of these students (see Figure 1a) to support them in developing a desirable understanding of slope and why. Additionally, we presented "The Hypothetical Student Situation" (adapted from Diamond, 2020; see Figure 1b), where a student questions the consistency of slope's meaning upon observing a function appearing steeper on one set of axes compared to another and solicited Ms. R's response to this confusion. To further explore Ms. R's nuanced understanding of slope as a composed unit ratio, we crafted follow-up questions. These questions were instrumental in highlighting the benefits of conceptualizing slope as a composed unit ratio.

Five students are discussing the meaning of slope in a linear context. Student A says that slope is $\frac{y_2-y_1}{x_2-x_1}$. Student B says that slope is the steepness of the line. Student C says that slope is rise over run. Student D says that slope is the rate of change of the line. Student E says that slope is the number m.



(a) (b)

Figure 1: (a) Five Students problem and (b) The Hypothetical Student Situation

Our analysis process involved a qualitative approach, initially conducting a conceptual analysis to understand Ms. R's verbal and non-verbal explanations, thereby constructing viable models of her mathematics (Steffe & Thompson, 2000). Our analysis relied on aforementioned characterizations of teachers' meanings for slope and meanings for ratio in order to identify Ms. R's meanings. We then attempted to connect these categories that we identified and seek potential implications of those meanings in Ms. R's classroom teaching.

Results

We structure our analysis around two main themes. Firstly, we identify Ms. R's meanings of slope and her concerns regarding traditional slope understandings, highlighting her preference for slope-as-ratio meaning. Secondly, we focus on the exploration of slope as a composed unit ratio, exemplified by Ms. R's pedagogical approach and its impact on student learning. Through this exploration, we offer evidence as to the potential effectiveness of viewing slope as a composed unit for fostering a deeper understanding of mathematical relationships.

Ms. R's Various Understanding of Slope

Analysis of the interviews suggested Ms. R's understanding of slope was multifaceted, encompassing several meanings: slope-as-steepness, slope-as-rise-over-run, slope-as-formula, slope-as-*m*, and slope-as-rate-of-change[1]. She also understood and articulated limitations of

[1] We adopt Ms. R's terminology, using "rate of change" to describe the "ratio" understanding of slope, despite our awareness of the conceptual differences between the two terms.

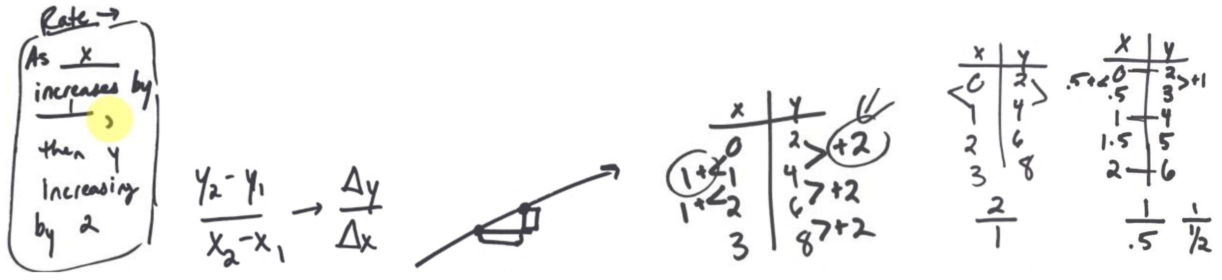
meanings for slope that did not include a rate of change understanding. For instance, in considering the Five Students problem (Figure 1a), Ms. R expressed concern about slope-as-steepness meaning in terms of its potential to mislead students into thinking that lines have the same slope based on their visual resemblance, regardless of axis scale or orientation. When asked The Hypothetical Student Situation task in which the same linear function is graphed on two coordinate axes with different scales (Figure 1b) and a student stated that the two functions had different slopes, she stated "steepness, that's probably where they're getting lost, because it'd be better to talk about change. I mean these lines, they don't initially look like they have the same slope…but they are the same graphs."

When considering slope as rise over run, Ms. R noted that students may rely on rise-over-run meanings because "that is their big middle school focus." Referencing the Five Students problem, she also suggested that students may rely on the slope formula only as a memorized fact: "this kid [Student A] spits out the slope formula because the teacher told him to over and over." Ms. R considered the slope formula to be more useful than the rise-over-run meaning due to its broader applicability to other scenarios, such as the arithmetic mean, and she saw both the rise-over-run meaning and the formula meaning to be superior to viewing slope as the number "m". Ms. R explained, "they know they are supposed to look at the number glued to the x [referring to $y = mx + b$], and that is it." Ms. R then wrote the formula $-2x + y = 3$, and explained that a student who viewed slope as the number "m" would get confused by an equation in this form: "They'll be like, what the heck happened in my graph, my equation?" Collectively, we had evidence from the interviews that Ms. R not only held these various meanings but was also able to position them against each other to discuss their productivity. Next, we illustrate her meaning of slope as a composed unit ratio.

Ms. R's Slope as Ratio Meaning: Composed Unit

Ms. R emphasized that she privileged Student D's meaning: "if this one actually knows what, like, rate of change of the line means, I like that one the best." She viewed Student D's meaning as versatile and applicable across various representations, such as table, equation, and graph, because it was not limited to specific formula or procedure. However, Ms. R's meaning was unclear. While she said she valued the rate-of-change meaning of slope, we were interested in exactly what her rate-of-change meaning was and, hence, what the meaning she valued involved. Did she construct a ratio as a multiplicative comparison, which she could view as a unit rate? Or did she have an alternate meaning for slope as a ratio or a rate of change? To gain insights into Ms. R's meanings, we asked her to create an example to describe how she would facilitate her students' development of the rate-of-change understanding. We hoped that by drawing attention to student development, we would not only gain insights into the meanings she could enact to solve problems but also those aspects of her meanings that she was consciously aware of.

Ms. R described an example with a speed of 25 miles per hour and explained that she could break this down into "for every 1 hour, the car is going 25 miles." Ms. R then explained that she could help students create similar phrases with a template: "As ___ increases/decreases by ___, then ___ increases/decreases by ___." Ms. R then clarified that students could use this template to generate equivalent ratios with different numbers, for instance, "for every two hours, the car is going 50 miles," which represents "the same slope." Ms. R's attention to the connection between the number of miles and the number of hours, combined with her understanding that this unit can be iterated to create other equivalent ratios, suggests a composed unit understanding of ratio.



(a) (b) (c) (d) (e)

Figure 2: (a) Slope as ratio, (b) Slope as formula, (c) Slope as rise over run, (d) Slope as rate of change in a table, (e) Determining the change in *y* for a 0.5-unit change in *x*.

Moreover, Ms. R did explicitly describe slope as a rate of change between two variables. She stated, "When I hear the word slope, I think rate of change…like two variables are changing." As she continued, her ongoing commentary suggested a composed unit ratio meaning: "as *x* increases by 1, *y* increases by 2." In order to demonstrate this meaning, Ms. R connected it to four representations (Figure 2a-d), pointing out that "they all connect, like they mean the same thing. But they just look different." Figure 2a shows the composed unit 2:1. Ms. R could also interpret this composed unit via the slope formula (Figure 2b), and she could also demonstrate that meaning visually on a graph that was not drawn to scale. In describing the table in Figure 2d, Ms. R noted, "it [referring to the change in *y*-values] goes up 2 for every 1 *x*."

Based on this response, we hypothesized that Ms. R's meaning of slope involved ratio as composed unit, but it was unclear whether she could also consider the slope ratio to be a multiplicative comparison between changes in quantities. We therefore pressed Ms. R on her examples in Figure 2, particularly about the meaning of the value of "2". Ms. R responded, "Technically, it's 2 over 1, but we like to simplify it to just 2 for some reason I don't know." We then asked her what "2" would mean if *x* changed by a value other than 1. With this, we aimed to determine if Ms. R understood the slope's value as an indicator of a multiplicative relationship, illustrating how many times the change in *y* is larger than the change in *x*. To answer our question, she created a new table (Figure 2e) and concluded that for a change in *x* of 0.5, the change in *y* should be 1. We took her actions and descriptions to suggest that Ms. R viewed her original ratio, 2:1, as a composed unit and partitioned it to create an equivalent ratio (1:0.5). This is a contraindication of slope as a multiplicative comparison as she did not multiply 0.5 by 2 to determine the change in *y*, nor did she ever appeal to a constant multiple between the two.

To further probe the extent Ms. R's slope-as-ratio meaning was consistent with a composed unit or multiplicative comparison, we also asked her to explain the division in the slope formula, i.e., to explain why one divides to calculate a slope such as 2/1 or 1/0.5. Ms. R struggled to provide a clear explanation, and she interpreted the vinculum (division bar) as a means for matching changes in quantities: "That is communicating 2 is matching with 1." Ms. R also explained, "to me, the division is this little comma [pointing to the comma in her phrase in Figure 2a]." These responses were further contraindications that Ms. R's meaning of slope-as-ratio entailed a multiplicative comparison. Despite our attempts to explore division in different contexts, she could not provide a substantial rationale beyond following the formula's instructions, remarking, "I'm trying to see how division comes in the way, and I'm not seeing it. I don't really know why I even divide, beside because the formula said so."

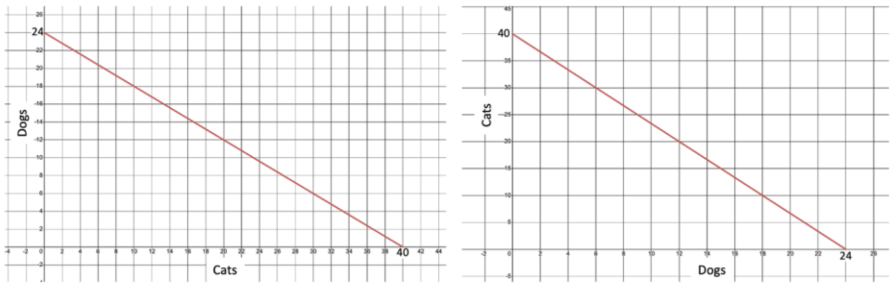
The Affordances and Constraints of Slope as Composed Unit

Ms. R's meaning of slope-as-ratio as a composed unit but not as a multiplicative comparison meant that she could develop equivalent ratios through iterating and partitioning the unit, but she could not directly determine the change in y-values for any associated change in x through multiplication. She also could not explain why slope involved division. This is consistent with the limitations identified in the literature, in which teachers struggle with conceptualizing slope and division as expressions of relative size, often viewing them non-quantitatively (Byerley & Thompson, 2017). These challenges hinder their ability to connect division with proportional reasoning (Coe, 2007), complicate the interpretation of points on graphs (Thompson, 2013), and hinder the recognition of slope as a consistent ratio of change (Stump, 2001).

Despite these limitations, we argue that there are also affordances to the meaning of slope as a composed unit ratio, beyond solving textbook problems. Ms. R could articulate meaningful connections based on a statement about slope as changes in y-values tied to corresponding changes in x-values; she could describe what that statement meant in terms of a function's graph, in terms of an associated table of values, and in terms of the formula for determining slope. Ms. R also had meaningful ways to support her own students in developing these connections. For instance, referring to the statement in Figure 2a and the graph in Figure 2c, she noted, "I would like for them to draw, like, the little triangle [drawing a triangle similar to that seen in Figure 2c], show me this part [referring to the run] and this part [pointing to the rise]."

Moreover, the meaning of slope as a composed unit ratio enabled Ms. R to conceive equivalence in the form of an invariant relationship across two graphs representing the same relationship in two different orientations (see Figure 3), which is a very sophisticated and productive way of thinking about graphical relationships (Moore et al., 2022). It also enabled her to respond in a meaningful way to a student thinking. We designed a task building on an activity she had implemented in the classroom, which referenced a pet-sitting business: A pet sitter can spend up to 8 hours each day feeding animals. Each cat requires 12 minutes per day, and each dog requires 20 minutes per day. We presented Ms. R with two graphs of the maximum numbers of dogs and cats that a pet sitter can feed, one with dogs on the x-axis and one with cats on the x-axis (see Figure 3), and we asked her which of the students' responses she agreed with, if any. Ms. R stated that Student D's response was the most correct, and therefore the slopes of the two graphs must be identical, despite their different visual representations and reciprocal values.

Five students are comparing the slopes of these graphs (see the figure below) in Pet Sitter Task context.



$d = 24 - \frac{3}{5}c$ $\qquad$ $c = 40 - \frac{5}{3}d$

Student A says that slope is $\frac{y_2 - y_1}{x_2 - x_1}$. She chooses two points in each graph and calculate the slope for each graph. So, the slope of graph on the left is $\frac{24-0}{0-40} = -\frac{24}{40} = -\frac{3}{5}$ and the slope of graph on the right is $\frac{40-0}{0-24} = -\frac{40}{24} = -\frac{5}{3}$. So, the slopes are different. In fact, the slopes are reciprocals of each other.

Student B says that slope is the steepness of the line. Both graphs look like they have the same steepness. Therefore, the slopes of these graphs are the same.

Student C says that slope is rise over run. Therefore, the slopes are different.

Student D says that slope is the rate of change. So, both graphs show that "For every 3 dogs you are done feeding, you can feed 5 more cats". So, the rate of change in both graphs are the same. Therefore, the slopes are the same.

[*Student D also knows that inputs can be represented on the y-axis and outputs can be represented on the x-axis*.**]**

Student E says that slope is the number *m*. So, the slopes are different.

Figure 3: Dog and Cat Feeding Time Functions and Graphs

We then presented Ms. R with another hypothetical student answer, which claimed that slope is a rate of change, but the two slopes were different because one graph showed that for every 5 cats fed you can feed 3 more dogs, and the other graph showed that for every 3 dogs fed you can feed 5 more cats. Therefore, the rates of change and thus slopes must be different. Ms. R maintained her original stance, arguing that the numerical values in the two slopes symbolized the same quantities, and thus represented the same rate of change. She elaborated, "I would be like, you said for every 5 cats, the variable after 5 is *cats* [emphasis added] in both of your sentences, you know, like, the variable after 3 is *dogs* [emphasis added]." Given that both expressions conveyed an equivalent meaning, Ms. R reasoned that "So, I would say, like, your rate of change is the same." She argued this was akin to reordering a sentence in different contexts. This highlights Ms. R's relational and quantitative understanding of slope viewed as a composed ratio. Her conceptualization enabled her to interpret the slopes of the two graphs as the same, and it allowed her to understand the meaning of slope in terms of coordinated changes in quantities as represented in tables, graphs, and equations.

Discussion and Conclusion

In this study, we have explored a nuanced meanings of slope as a ratio, proposing a new perspective that transcends the algebraic or geometric ratio conceptions identified by Nagle and Moore-Russo (2013), and yet does not reach the level of multiplicative comparison. While the multiplicative comparison conception of slope is recognized for its depth of understanding, it remains a challenging achievement for both teachers and students. This insight aligns with the broader literature, which has long noted the difficulties inherent in grasping slope from a multiplicative standpoint (Cho & Nagle, 2017; DeJarnette et al., 2020; Dolores-Flores et al., 2020). A potential implication of our research is that understanding ratio as a composed unit may serve as a more accessible—since it is not as cognitively complex as a multiplicative meaning—and meaningful foundation for developing deeper insights into the concept of slope.

The case of Ms. R illustrates that benefits traditionally associated with understanding slope as a multiplicative comparison are still attainable through the lens of a composed unit ratio. Ms. R's slope meaning allowed a fully quantitative understanding, enabling her to effectively interpret and apply the concept of slope in various contexts. She correctly used algebraic manipulations to determine slopes and connected a slope value to the constant rate of change in a linear function (Cho & Nagle, 2017; DeJarnette et al., 2020; Diamond, 2020; Dolores-Flores et al., 2020; Lobato & Siebert, 2002; Talib et al., 2023). This underscores the flexibility and effectiveness of the composed unit ratio approach in fostering a comprehensive understanding of slope.

To clarify, our stance is not to undermine the importance of understanding ratios as multiplicative comparisons or their relevance in understanding the concept of slope. Instead, we propose prioritizing the development of an understanding of ratios as composed units as a foundational step. This strategy involves encouraging the development of equivalent slopes as ratios of changes that leverage values smaller than 1 and values with "messy numbers," thereby enhancing learners' understanding of the invariant relationship between changes in *y*-values and their corresponding *x*-values. For instance, Ms. R's ability to conceptualize an equivalent slope of 1/0.5 from an initial slope of 2/1 exemplifies the potential of this method to deepen understanding. We could ask similar questions to encourage other equivalent slopes, such as 14/7, 9/4.5, 0.5/0.25, or 0.4/0.2. Asking learners to reflect on what is invariant across all these different ratios could encourage attention to the fact that regardless of the increase in *y*-values, the increase in *x*-values remain twice as large.

We must acknowledge that our insights are based on the experiences of a single teacher, Ms. R, providing a compelling case that it is feasible to conceptualize slope as a ratio in effective and impactful ways without necessarily incorporating the notion of multiplicative comparison. Ms. R's example shines a light on the viability of comprehending slope through the lens of a composed unit. However, further research is needed to assess the prevalence and efficacy of this approach among broader populations, including pre-service teachers and secondary students. We believe that such investigations will contribute significantly to the mathematics education field by offering alternative pathways to understanding slope, thereby enriching pedagogical strategies and student learning experiences.

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